Lecture 21

- Intersection form of G/P, G/B] actually covered
- Coinvariants presentation of cohomology.] planned, didn't do.

$w_0 \in W$ longest element. $w_0 \Phi^+ = \Phi^-$. $\ell(w_0) = \dim G/B$.

($w_0 W_P$ is the longest coset.)

Schubert basis of $H^*(G/P)$: $\{[X_{aW_P}] \mid aW_P \in W/W_P\}$ ← commutative graded ring.

where $[X_{aW_P}]$ is in degree $2\ell(aW_P)$.

<u>Intersection form</u>. Let $x, y \in H^*(X)$ be in compl. degrees (X compact, connected, oriented). So $\deg(x) + \deg(y) = \dim X$.

$\langle x, y\rangle = k$ if $xy = k \cdot \mu$ where $\mu \in H^{\dim X}(X)$ is the gen. (Poincaré dual to a point)

$\langle , \rangle$ is the <u>intersection pairing</u> or <u>int form</u>.

Why? fund thm of int theory.

<u>Thm</u> Let M, N be compact oriented submflds of X with $\dim M = m$, $\dim N = n$, $m + n = \dim X =: d$. Spse that $\forall p \in M \cap N$, $T_p X = T_p M \oplus T_p N$. Then

(say p is a point of <u>transverse intersection</u>)

$$\langle [M], [N] \rangle = \sum_{p \in M \cap N} \varepsilon_p$$

fund class, in $H^{d-m}(X)$ or $H^{d-n}(X)$

where $\varepsilon_p = 1$ if $(\underbrace{u_1, \dots, u_m}_{+\text{ basis } T_pM}, \underbrace{v_1, \dots, v_n}_{+\text{ basis } T_pN})$ is oriented basis of $T_p X$.

-1 else.

When M, N, X complex with induced orient, $\varepsilon_p = 1$ $\forall p \in M \cap N$.

So if $[X_{aw_P}]$ etc were smooth, you would think

$$\langle [X_{aw_P}], [X_{bw_P}] \rangle = \#\text{ int points of } X_{aw_P} \text{ and } X_{bw_P}.$$

(use X_{aw_P} as C_{aw_P} not compact!) in case $\ell(aW_P) + \ell(bW_P) = \ell(w_0 W_P)$

Problem: Definitely not transverse!
often not smooth. (no ex so far...)

Irred (alg-geom "connected") proj varieties have a dimension. Can be defined alg. but is equiv to:

0-dim: point

k-dim: Can remove a finite union of (k-1)-dim irred proj var to obtain an embedded submfld of $\mathbb{CP}^N$ of $\mathbb{C}$ dim k.

There's a well-defined smallest set you can remove, and then the rest is called the <u>smooth locus</u>.

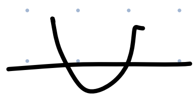
There's an algebraic formulation of intersection theory (χ(Tor...)). Just like there being an alg notion of "fundamental class".

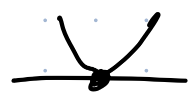
Here's a consequence.

<u>Thm</u>. Suppose $M, N \subset X$ all irred proj var, $\dim M + \dim N = \dim X$. Suppose $M \cap N \subset M^{sm} \cap N^{sm}$ and all such int are transverse. Then $\langle [M], [N] \rangle = \#(M \cap N)$. □

see Hartshorne III.

Actually the situation is nicer. If the int points are in $M^{sm} \cap N^{sm}$ and are isolated, you can assign a "multiplicity" to make this work.


transverse


Not trans.
Mult = 2.

ASIDE – not used

Thm (Kleiman 1974): Let X_a, X_b be of compl dim in G/P. There is an open dense subset of G (Zariski open in fact) s.t. $X_a \cap gX_b = C_a \cap gC_b$ and consists of transverse int.

Now G conn so $[gX_b] = [X_b]$. (They're htpic in G/P!)

So we're reduced to knowing about $X_a \cap_g X_b$ for such g.

Opposite cells. Let $X^a = w_0 X_{w_0 a}$.

Just as X_a is a B-orbit closure, X_b is a $B^- = w_0 B w_0$ orbit closure.

$$X_a = BaP \qquad X^a = w_0 B w_0 a P = B^- a P.$$

Now X_a and X^a obviously intersect at aP. Also, they have complementary dimension. Also $[X^a] = [X_{w_0 a}]$

Opp cells thm -

Thm. $X_a \cap X^a = aP$ and it is a transverse int of smooth pts. If a, b are distinct elements of the same length then $X_a \cap X^b$ empty.

Cor. $\langle [X_a], [X^b] \rangle = \delta_{ab}$

or $\langle [X_a], [X_b] \rangle = \begin{cases} 1 & a = w_0 b \\ 0 & \text{else.} \end{cases}$

Idea for G/B. $\mathcal{U} = \exp\left(\bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha\right)$ $\quad \mathcal{U}^- = w_0 \mathcal{U} w_0$.

X_a is $\mathcal{U}$-orbit of aB $\qquad X^a$ is $\mathcal{U}^-$ orbit of aB.

Under $dL_{a^{-1}} : T_{aB} G/B \longrightarrow \mathfrak{g}/\mathfrak{b} \cong \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$,

$T_{aB} X_a$ corresp to $\alpha \in \Phi_a^+$ (the a-flipped ones)

$T_{aB} X^a$ corresp to $\alpha \in \Gamma_a^+$ (the a-non-flipped ones)

Transverse as $\Phi_a^+ \cap \Gamma_a^+ = \phi$.

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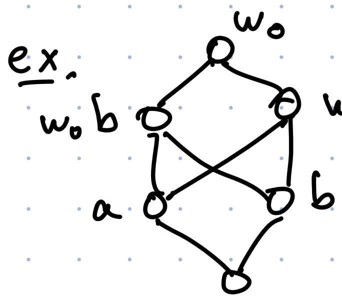
ex.



3	0
2	1
1	2
0	3

$$\langle [X_e], [X_{w_0}] \rangle = \langle [X_a], [X_{w_0 a}] \rangle = \langle [X_b], [X_{w_0 b}] \rangle = 1$$

($[X_{w_0 a}] = [X^a]$)

all other int zero.

Ex. Here $X_a = \{(p, \ell) \mid \ell = [e_1, e_2]\}$ $\quad X_{w_0 a} = ?$ $\quad X_a \cap X^a = ?$

Borel's Description

$G/B \cong K/T$ where K cpt real form, $T = B \cap K = H \cap K \cong (S^1)^n$

e.g. $SL_n\mathbb{C}/B \cong SU(n)/S(U(1)^n)$.

Let $\mathcal{P}$ = Sym tensor alg in $\mathfrak{t}^*$, $\mathfrak{t} = \mathrm{Lie}(T)$

i.e. if $x_1, \dots, x_r : \mathfrak{t} \to \mathbb{R}$ basis of $\mathfrak{t}^*$ then $\mathcal{P} \cong \mathbb{R}[x_1, \dots, x_r]$

Then W acts on $\mathfrak{t}^*$ by adjoint of Ad action on $\mathfrak{t}$.

So $W \curvearrowright \mathcal{P}$. Let $\mathcal{P}^W$ denote the invts.

J = the ideal of $\mathcal{P}$ generated by $(\mathcal{P}^W)_+$, the pos degree elts.

<u>Thm (Borel)</u> There is a degree-doubling iso

$$\mathcal{P}/J \longrightarrow H^*(K/T) \cong H^*(G/B).$$